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Abstract

Thesis title: Gain-Scheduled Controller Design

Keywords: Gain-scheduled control; Lyapunov theory of stability; Guaranteed cost control; Bellman-Lyapunov function; LPV system; Robust control; Input/output constraints

This thesis is devoted to controller synthesis, i.e. to finding a systematic procedure to determine the optimal (sub-optimal) controller parameters which guarantees the closed-loop stability and guaranteed cost for uncertain nonlinear systems with considering input/output constraints, all this without on-line optimization. The controller in this thesis is given in a feedback structure, that is the controller has information about the system and uses this information to influence the system. In this thesis the linear parameter-varying based gain scheduling is investigated. The nonlinear system is transformed to a linear parameter-varying system, which is used to design a controller, i.e. a gain-scheduled controller with consideration of the objectives on the system. The gain-scheduled controller synthesis in this thesis is based on the Lyapunov theory of stability as well as on the Bellman-Lyapunov function. Several forms of parameter dependent/quadratic Lyapunov functions are presented and tested. To achieve performance quality a quadratic cost function and its modifications known from LQ theory are used. In this thesis one can find also an application of gain scheduling in switched and in model predictive control with consideration of input/output constraints. The main results for controller synthesis are in the form of bilinear matrix inequalities (BMI) and/or linear matrix inequalities (LMI). For controller synthesis one can use a free and open source BMI solver PenLab or LMI solvers LMILab or SeDuMi. The synthesis can be done in a computationally tractable and systematic way, therefore the linear parameter-varying based gain scheduling approach presented in this thesis is a worthy competitor to other controller synthesis methods for nonlinear systems.

Anotácia dizertačnej práce

Názov dizertačnej práce: Riadenie systémov metódou "gain scheduling"

Kľúčové slová: Riadenie s plánovaným zosilnením; Lyapunová teória stability; Riadenie s garantovanou kvalitou; Bellman-Lyapunová funkcia; LPV systémy; Robustné riadenie; Vstupné/výstupné obmedzenia

Táto práca sa venuje problematike návrhu regulátora, tj. nájsť systematický postup na návrh optimálnych (suboptimálnych) parametrov regulátora, ktoré garantujú stabilitu a kvalitu v uzavretej slučke, pri obmedzení vstupno-výstupných hodnôt systémov pre nelineárne systémy s neurčitosťami, a to bez on-line optimalizácie. Uvedený regulátor má spätno-väzobnú riadiacu štruktúru, čo znamená, že disponuje informáciami o danom systéme, ktoré využíva k jeho ovplyvneniu. Táto práca sa podrobnejšie zaoberá s riadením s plánovaným zosilnením, a to na báze parametricky závislých lineárnych systémov. Nelineárny systém je pretransformovaný na parametricky závislý lineárny systém, čo sa následne využíva na návrh regulátora, tj. regulátora s plánovaným zosilnením, s ohľadom na požiadavky daného systému. Syntéza regulátora s plánovaným zosilnením sa uskutoční na báze Lyapunovej teórie stability s použitím Bellman-Lyapunovej funkcie, v rámci čoho sú prezentované a testované rôzne typy kvadratickej a parametricky závislej Lyapunovej funkcie. Pre dosiahnutie požadovanej kvality sa používa kvadratická účelová funkcia známa z LQ riadenia, s rôznymi modifikáciami. V tejto práci nájdeme aj aplikáciu riadenia s plánovaným zosilnením v oblasti takzvaného prepínacieho riadenia (switched control), ako aj v rámci prediktívneho riadenia (model predictive control). Hlavné výsledky pre syntézu regulátorov sú v tvare bilineárnych maticových nerovníc (BMI) a/alebo linearných maticových nerovníc (LMI). Na návrh regulátorov môžeme používať bezplatný a „open source“ BMI solver PenLab alebo LMI solvre LMILab a SeDuMi. Uvedené skutočnosti umožnia vykonať syntézu jednoduchým a systematickým spôsobom. Riadenie s plánovaným zosilnením na báze parametricky závislých lineárnych systémov prezentované v tejto práci je vhodným konkurentom vo vzťahu k iným metódam syntézy regulátorov pre nelineárne systémy.

Contents

1 Introduction

This thesis is devoted to controller synthesis, i.e. to finding a systematic procedure to determine the optimal (sub-optimal) controller parameters which guarantees the closed-loop stability and guaranteed cost for uncertain nonlinear systems with considering input/output constraints. In consideration of the objectives stated for the system such as tracking a reference signal or keeping the plant at a desired working point (operation point) and based on the knowledge of the system (plant), the controller takes decisions. The controller in this thesis is given in a feedback structure, which means that the controller has information about the system and uses it to influence the system. A system with a feedback controller is said to be a closed-loop system.

To design a controller which satisfies the objectives we need an adequately accurate model of the physical system. Nevertheless, real plants are hard to describe exactly. Alternatively, the designed controller must handle the cases when the state of the real plant differs from what is observed by the model. A controller that is able to handle model uncertainties and/or disturbances is said to be robust, and the theory dealing with these issues is said to be robust control.

The robust control theory is well established for linear systems but almost all real processes are more or less nonlinear. If the plant operating region is small, one can use the robust control approaches to design a linear robust controller, where the nonlinearities are treated as model uncertainties. However, for real nonlinear processes, where the operating region is large, the above mentioned controller synthesis may be inapplicable because the linear robust controller may not be able to meet the performance specifications. For this reason the controller design for nonlinear systems is nowadays a very determinative and important field of research.

Gain scheduling is one of the most common used controller design approaches for nonlinear systems and has a wide range of use in industrial applications. Many of the early articles were associated with flight control and aerospace. Then, gradually, this approach has been used almost everywhere in control engineering, which was greatly advanced with the introduction of LPV systems.

Linear parameter-varying systems are time-varying plants whose state space matrices are fixed functions of some vector of varying parameters $\theta(t)$. They were introduced first by Jeff S. Shamma in 1988 to model gain scheduling. Today the LPV paradigm has become a standard formalism in systems and controls with lots of researches and articles devoted to analysis, controller design and system identification of these models.

This thesis deals with linear parameter-varying based gain scheduling, which means that the nonlinear system is transformed to a linear parameter-varying system, which is used to design a controller, i.e. gain-scheduled controller. The problem formulation is close to the linear system counterpart, therefore using LPV models to design a controller has potential computational advantages over other controller synthesis methods for nonlinear systems. Not to mention that the LPV based gain scheduling approaches comes with a theoretical validity because the closed-loop system can meet certain specifications. Nonetheless, following the literature it is ascertainable that there are still many unsolved problems. This thesis is devoted to some of these problems.

1.1 Goals & Objectives

As already mentioned, there are many unsolved problems. Therefore, it is necessary to find new and novel controller design approaches. If one wants to summarize the main goal of this thesis in one sentence, then you would read: The main goal is to find a controller design approach for uncertain nonlinear systems, which guarantees the closed-loop stability and guaranteed cost with considering input/output constraints, all this without on-line optimization and need of high-performance industrial computers. That is why we set the following goals:

- Suggest a gain-scheduled PID controller design approach with guaranteed cost in continuous and discrete time state space using BMI
- Suggest a robust gain-scheduled PID controller design approach with guaranteed cost and parameter dependent quadratic stability in state space using BMI
- Suggest a variable weighting gain-scheduled approach
- Convert some BMI controller design approaches to LMI
- Suggest a switched and model predictive gain-scheduled method
- Suggest a gain-scheduled controller design approach with input/output constraints
- Apply methods to relevant processes

1.2 Outline

The sequel of this summary of dissertation thesis is organized as follows. In *Section 2* one can find a preliminary chapter, where with review of the literature a brief overview of linear parameter-varying systems and gain scheduling are presented. In *Section 3-5* one can find an overview of selected research results which covers the main research results obtained within the last 2.5 years . Finally, in *Section 6*, following the selected results from selected papers, some concluding remarks and suggestions for future research are given.

2 Preliminary pages

In this section preliminaries of linear parameter-varying systems as well as gain scheduling are introduced. This section is intended to highlight the properties and give a short background to the tools used in the appended papers.

2.1 Linear parameter-varying systems

Linear parameter-varying systems are time-varying plants whose state space matrices are fixed functions of some vector of varying parameters $\theta(t)$. It was introduced first by Jeff S. Shamma in 1988 [1] to model gain scheduling. *"Today LPV paradigm has become a standard formalism in systems and controls with lot of researches and articles devoted to analysis, controller design and system identification of these models"*, as Shamma wrote in [2]. This section deals with LPV models and presents analytical approaches for LPV systems.

2.1.1 Introduction to LPV systems

Linear parameter-varying systems are time-varying plants whose state space matrices are fixed functions of some vector of varying parameters $\theta(t)$. Linear parameter-varying (LPV) systems have the following interpretations:

- they can be viewed as linear time invariant (LTI) plants subject to time-varying known parameters $\theta(t) \in \langle \underline{\theta}, \overline{\theta} \rangle$,
- they can be models of linear time-varying plants,
- they can be LTI plant models resulting from linearization of the nonlinear plants along trajectories of the parameter $\theta(t) \in \langle \underline{\theta}, \overline{\theta} \rangle$ which can be measured.

For the first and third class of systems, parameter θ can be exploited for the control strategy to increase the performance of closed-loop systems. Hence, in this thesis the following LPV system will be used:

$$\begin{aligned} \dot{x} &= A(\theta(t))x + B(\theta(t))u \\ y &= Cx \end{aligned} \tag{1}$$

where for the affine case

$$\begin{aligned} A(\theta(t)) &= A_0 + A_1\theta_1(t) + \ldots + A_p\theta_p(t) \\ B(\theta(t)) &= B_0 + B_1\theta_1(t) + \ldots + B_p\theta_p(t) \end{aligned}$$

and $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is a control input, $y \in \mathbb{R}^l$ is the measurement output vector, A_0, B_0, A_i, B_i, $i = 1, 2 \ldots, p$, C are constant matrices of appropriate dimension, $\theta(t) \in \langle \underline{\theta}, \overline{\theta} \rangle \in \Omega$ and $\dot{\theta}(t) \in \langle \underline{\dot{\theta}}, \overline{\dot{\theta}} \rangle \in \Omega_t$ are vectors of time-varying plant parameters which belong to the known boundaries.

The LPV paradigm was introduced by Jeff. S. Shamma in his Ph.D. thesis [1] for the analysis of gain-scheduled controller design. The authors in early works (see [1, 3, 4, 5, 6, 7, 8] and surveys [9, 10]) in gain scheduling the LPV system framework called as the golden mean between linear and nonlinear dynamics, because "*the LPV system is an indexed collection of linear systems, in which the indexing parameter is exogenous, i.e., independent of the state.*"(wrote J. S.

Shamma in his Ph.D. thesis [1]). In gain scheduling, this parameter is often a function of the state, and hence endogenous

$$\begin{aligned}\dot{x} &= A(z)x + B(z)u \\ y &= C(z)x \\ z &= h(x)\end{aligned} \tag{2}$$

2.1.2 Application of the LPV systems

Since the first publication devoted to LPV systems, the LPV paradigm has been used in several fields in control engineering including the modeling and control design. Traditionally the gain scheduling was the primary design approach for flight control and consequently many of the first articles and papers which applied and improved the LPV framework were associated with flight control. Afterwards continuously many papers and articles have appeared which are using LPV paradigm in several application areas such as flight control and missile autopilots [11, 12, 13, 14, 15, 16, 17], aeroelasticity [18, 19, 20, 21], magnetic bearings [22, 23, 24, 25], automotive bearings [26, 27, 28], energy and power systems [29, 30, 31, 32, 33, 34], turbofan engines [35, 36, 37, 38], microgravity [39, 40, 41], diabetes control [42, 43, 44], anesthesia delivery [45], IC manufacturing [46].

Due to the success of LPV paradigm in 2012 for the twentieth anniversary of the invention of LPV paradigm a gift edition book was published by Javad Mohammadpour and Carsten W. Scherer Editors at Springer [2] which is fully devoted to LPV systems.

2.2 Gain scheduling

The robust control theory is well established for linear systems but almost all real processes are more or less nonlinear. If the plant operating region is small, one can use the robust control approaches to design a linear robust controller where the nonlinearities are treated as model uncertainties. However, for real nonlinear processes, where the operating region is large, the above mentioned controller synthesis may be inapplicable. For this reason the controller design for nonlinear systems is nowadays a very determinative and important field of research.

Gain scheduling is one of the most common used controller design approaches for nonlinear systems and has a wide range of use in industrial applications. In this section the main principles, several classical approaches and finally the linear parameter-varying based version of gain scheduling are presented and investigated.

2.2.1 Introduction to gain scheduling

In literature a lot of term are meant under gain scheduling (GS). For example switching or blending of gain values of controllers or models, switching or blending of complete controllers or models or adapt (schedule) controller parameters or model parameters according to different operating conditions. A common feature is the sense of decomposing nonlinear design problems into linear or nonlinear sub-problems. The main difference lies in the realization.

Consequently gain scheduling may be classified in different way

- According to decomposition
  1. GS methods decomposing nonlinear design problems into linear sub-problems
  2. GS methods decomposing nonlinear design problems into nonlinear (affine) sub-problems
- According to signal processing
  1. Continuous gain scheduling methods
  2. Discrete gain scheduling methods
  3. hybrid or switched gain scheduling methods
- According to main approaches
  1. Classical (linearization based) gain scheduling
  2. LFT based GS synthesis
  3. LPV based GS synthesis
  4. Fuzzy GS techniques
  5. Other modern GS techniques

2.2.2 History of gain scheduling

The ferret in the history of gain scheduling appears in the 1960s but a similar simpler technique was used in World War II toat control the rockets V2 (switching controllers based on measured data). It is not surprising therefore that gain scheduling as a concept or notion firstly appear in flight control and later in aerospace. Leith and Leithead in their survey [9] and likewise also Rugh and Shamma in their survey paper [10] considered the first appearance of GS from the 1960s. Rugh stated in his survey that "*Gain control*" does appear in the 25th Anniversary Index (1956–1981) published in 1981 but only one of the five listed papers is relevant to gain scheduling. Also *Automatica* lists gain scheduling as a subject in its 1963–1995 cumulative index published in 1995. Of the four citations given, only one dated earlier than 1990 [1]. It can be stated that increased attention to gain scheduling appeared after introducing the LPV paradigm by

Jeff. S. Shamma (1988). This is partly understandable because LPV paradigm allowed to describe nonlinear system as a family of linear systems and hence investigate the stability of these systems. Figure 1 shows the major dates with remarks in a time-line of gain scheduling.

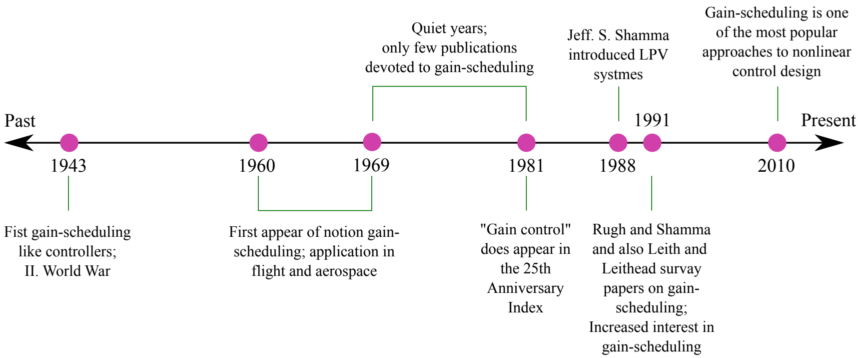

Figure 1: The time-line of gain scheduling

2.2.3 Application of gain scheduling

As already noted, traditionally the gain scheduling was the primary design approach to flight control and, consequently, many of the first articles and papers were associated with flight control [47, 48, 49, 50, 51, 52, 53, 54] and aerospace [55, 56, 57]. Then gradually GS has been used almost everywhere in control engineering which was greatly advanced with the introduction of LPV systems.

The second big bang in the history of gain scheduling was the advent of fuzzy gain scheduling. Today, every second paper that appears under gain scheduling is devoted to fuzzy gain scheduling. Due to this wide range of gain-schedule approaches, gain scheduling is now used in several fields in practice. For example in power systems the gain scheduling enjoyed exceptional success in control of wind turbines [58, 59, 60, 61, 62, 63, 64]. But beside all this, some papers are devoted to hydro turbines [65, 66], gas turbines [67], power system stabilizers [68] and generators [69]. Many papers in gain scheduling are devoted to magnetic bearings [70, 71, 72, 73, 74, 75] but we can find some papers devoted to also to microgravity [76], turbofan engine [77] and diabetes control [78].

2.2.4 Summary of gain scheduling

The main advantage of classical gain scheduling is that it inherits the benefits of linear controller design methods, including intuitive classical design tools and time as well as frequency domain performance specifications. PID control is the most used control strategy in industrial applications due to its relatively simple and intuitive design, hence this is a major advantage with respect to other nonlinear controller design syntheses. The approach thus enables the design of low computational effort controllers. Conceptually, gain scheduling involves an intuitive simplification of the problem into parallel decompositions of the total system.

LPV and LFT synthesis require a true LPV model as a basis. In general however, gain scheduling may be employed in the absence of an analytical model, e.g. on the basis of a collection of plant linearizations. Consequently, controller design based on a whitebox as well as a blackbox and even data-based 'modeling' is possible. If the possibility of fast parameter variations is not addressed in the design process, guaranteed properties of the overall gain-scheduled design cannot be established. The main advantage of LPV and LFT control synthesis is that they do account for parameter variations in the controller design, which results in a priori guarantees regarding stability and performance specifications. The main drawback of LPV and LFT control synthesis involves conservativeness, which has to be introduced to enable solving the resulting LMIs. As a result of that, current LPV and LFT syntheses comprise specific extensions of robust control techniques rather than true generalizations. However, current and future research still provides and will provide less conservative solutions.

The main drawback of fuzzy gain scheduling involves the lack of a relation between the dynamic characteristics of the original nonlinear model and the fuzzy model. Even locally, the dynamics of the fuzzy model can not be related to the original nonlinear model. Fuzzy gain scheduling techniques may involve classical gain scheduling alike as well as LPV techniques.

The analysis and theorems stated herein are presented in an informal manner. Technical details may (and should) be found in the associated references.

2.3 Discussion

This thesis is devoted to gain scheduling within this to LPV based gain scheduling because in our opinion the biggest potential between gain scheduling approaches is in the LPV based gain scheduling. Despite this, we described all main historical approaches to gain scheduling as classical gain scheduling, LFT based gain scheduling and novel fuzzy gain scheduling.

As we mentioned LPV based gain scheduling appear in 1988 when Jeff. S. Shamma introduced the LPV paradigm in his Ph.D. thesis [1]. Today LPV paradigm has become a standard formalism in systems and controls with lots of researches and articles devoted to analysis, controller design and system identification of these models. Due to this nowadays the LPV gain scheduling belongs to the most popular approaches to nonlinear control design. But, as we mentioned in *Introduction*, there are still a lot of unsolved problems. Browsing through literature we cannot find any general LPV based gain-scheduled approach which will involve guaranteed cost and affine quadratic stability. In addition there are very rudimentary approaches in switched and predictive control not to mention the robust and discrete design approaches.

Currently, nowhere it is solved how to affect the performance quality separately in each working point when direct LPV controller approach is used. Furthermore, there are only few papers devoted to output feedbacks and they also not use fixed order output feedbacks like PID/PSD controllers.

Most of the papers devoted to LPV based gain scheduling convert the stability conditions into LMI problems. But currently we cannot find a general LMI approach with guaranteed stability and guaranteed cost. Furthermore, nowhere it is solved how to consider input/output constraints without need of on-line optimization.

Among other things, to find a solution for some of these unsolved problems (deficiencies) were the main goals of the research which is summarized in this thesis.

3 Robust Gain-Scheduled PID Controller Design for Uncertain LPV Systems

A novel methodology is proposed for robust gain-scheduled PID controller design for uncertain LPV systems. The proposed design procedure is based on the parameter-dependent quadratic stability approach. A new uncertain LPV system model has been introduced in this paper. To access the performance quality, the approach of a parameter-varying guaranteed cost is used which allowed to reach the desired performance for different working points. Several forms of parameter dependent quadratic stability are presented which withstand arbitrarily fast model parameter variation or/and arbitrarily fast gain-scheduled parameter variation.

3.1 Problem formulation and preliminaries

Consider a continuous-time linear parameter-varying (LPV) uncertain system in the form

$$\begin{aligned} \dot{x} &= \overline{A}(\xi,\theta)x + \overline{B}(\xi,\theta)u \\ y &= Cx \\ \dot{y}_d &= C_d\dot{x} \end{aligned} \tag{3}$$

where linear parameter-varying matrices

$$\begin{aligned} \overline{A}(\xi,\theta) &= A_0(\xi) + \sum_{i=1}^{s} A_i(\xi)\theta_i \ \in \mathbb{R}^{n\times n} \\ \overline{B}(\xi,\theta) &= B_0(\xi) + \sum_{i=1}^{s} B_i(\xi)\theta_i \ \in \mathbb{R}^{n\times m} \end{aligned} \tag{4}$$

$x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^l$ denote the state, control input and controlled output, respectively. Matrices $A_i(\xi)$, $B_i(\xi)$, $i = 0,1,2,\ldots,s$ belong to the convex set:

a polytope with N vertices that can be formally defined as

$$\Omega = \left\{ A_i(\xi),\ B_i(\xi) = \sum_{j=1}^{N} (A_{ij}, B_{ij})\, \xi_j \right\}, \qquad i = 0, 1, 2, \ldots, s, \quad \sum_{j=1}^{N} \xi_j = 1, \quad \xi_j \geq 0 \tag{5}$$

where s is the number of scheduled parameters; ξ_j, $j = 1, 2, \ldots, N$ are constant or possibly time-varying but unknown parameters; matrices A_{ij}, B_{ij}, C, C_d are constant matrices of corresponding dimensions, where C_d is the output matrix for D part of the controller. $\theta \in \mathbb{R}^s$ is a vector of known (measurable) constant or possibly time-varying scheduled parameters. Assume that both lower and upper bounds are available. Specifically

1. Each parameter θ_i, $i = 1, 2, \ldots, s$ ranges between known extremal values

$$\theta \in \Omega_s = \left\{ \theta \in \mathbb{R}^s : \ \theta_i \in \langle \underline{\theta}_i, \overline{\theta}_i \rangle,\ i = 1, 2, \ldots, s \right\} \tag{6}$$

2. The rate of variation $\dot{\theta}_i$ is well defined at all times and satisfies

$$\dot{\theta} \in \Omega_t = \left\{ \dot{\theta}_i \in \mathbb{R}^s : \ \dot{\theta}_i \in \langle \underline{\dot{\theta}}_i, \overline{\dot{\theta}}_i \rangle,\ i = 1, 2, \ldots, s \right\} \tag{7}$$

Note that system (3), (4), (5) consists of two type of vertices. The first one is due to the gain-scheduled parameters θ with $T = 2^s$ vertices – θ vertices, and the second set of vertices are due to uncertainties of the system – N, ξ vertices. For robust gain-scheduled "I" part controller design the states of system (3) need to be extended in such a way that a static output feedback control algorithm can provide proportional (P) and integral (I) parts of the designed controller. For more details see [79]. Assume that system (3) allows PI controller design with a static output feedback.

To access the system performance, we consider an original scheduling quadratic cost function

$$J = \int_0^\infty J(t)dt = \int_0^\infty \left(x^T Q(\theta) x + u^T R u + \dot{x}^T S(\theta) \dot{x} \right) dt \tag{8}$$

where

$$Q(\theta) = Q_0 + \sum_{i=1}^{s} Q_i \theta_i, \quad S(\theta) = S_0 + \sum_{i=1}^{s} S_i \theta_i$$

The feedback control law is considered in the form

$$u = F(\theta) y + F_d(\theta) \dot{y}_d \tag{9}$$

where

$$F(\theta) = F_0 + \sum_{i=1}^{s} F_i \theta_i, \quad F_d(\theta) = F_{d0} + \sum_{i=1}^{s} F_{di} \theta_i$$

Matrices $F_i,\ F_{di},\ i=0,1,2,\dots,s$ are the static output PI part and the output derivative feedback gain-scheduled controller. The structure of the above matrices can be prescribed.

The respective closed-loop system is then

$$M_d(\xi,\theta)\dot{x} = A_c(\xi,\theta)x \tag{10}$$

where

$$M_d(\xi,\theta) = I - \overline{B}(\xi,\theta)F_d(\theta)C_d$$
$$A_c(\xi,\theta) = \overline{A}(\xi,\theta) + \overline{B}(\xi,\theta)F(\theta)C$$

Let as recall some results about an optimal control of time-varying systems [80].

Lemma 1. *Let there exists a scalar positive definite function $V(x,t)$ such that $\lim_{t\to\infty} V(x,t) = 0$ which satisfies*

$$\min_{u\in\Omega_u}\left\{\frac{\delta V}{\delta x}A_c(\theta) + \frac{\delta V}{\delta t} + J(t)\right\} = 0 \tag{11}$$

From (11) *obtained control algorithm $u = u^*(x,t)$ ensure the closed-loop stability and on the solution of* (3) *optimal value of cost function as $J^* = J(x_0,t_0) = V(x(0),t_0)$.*

Eq. (11) is known as Bellman-Lyapunov equation and function $V(x,t)$ which satisfies to (11) is Lyapunov function. For a given concrete structure of Lyapunov function the optimal control algorithm may reduces from "*if and only if*" to "*if*" and for switched systems, robust control, gain-scheduled control and so on to guaranteed cost.

Definition 1. Consider a stable closed-loop system (10). If there exists a control law u (9) which satisfies (13) and a positive scalar J^* such that the value of closed-loop cost function (8) J satisfies $J < J^*$ for all $\theta \in \Omega_s$ and ξ_j, $j = 1,2\dots,N$ satisfying (5), then J^* is said to be a guaranteed cost and u is said to be a guaranteed cost control law for system (10).

Let us recall some parameter dependent stability results which provide basic further developments.

Definition 2. Closed-loop system (10) is parameter dependent quadratically stable in the convex domain Ω given by (5) for all $\theta \in \Omega_s$ and $\dot{\theta} \in \Omega_t$ if and only if there exists a positive definite parameter dependent Lyapunov function $V(\xi,\theta)$ such that the time derivative of Lyapunov function with respect to (10) is

$$\frac{dV(\xi,\theta,t)}{dt} < 0 \tag{12}$$

Lemma 2. *Consider the closed-loop system* (10). *Control algorithm* (9) *is the guaranteed cost control law if and only if there exists a parameter dependent Lyapunov function $V(\xi,\theta)$ such that the following condition holds [80]*

$$B_e(\xi,\theta) = \min_u\left(\frac{dV(\xi,\theta,t)}{dt} + J(t)\right) \le 0 \tag{13}$$

Uncertain closed-loop system (10) conforming to *Lemma 2* is called robust parameter dependent quadratically stable with guaranteed cost.

We proceed with the notion of multi-convexity of a scalar quadratic function [81].

Lemma 3. *Consider a scalar quadratic function of* $\theta \in \mathbb{R}^s$

$$f(\theta) = \alpha_0 + \sum_{i=1}^{s} \alpha_i \theta_i + \sum_{i=1}^{s} \sum_{j>i}^{s} \beta_{ij} \theta_i \theta_j + \sum_{i=1}^{s} \gamma_i \theta_i^2 \tag{14}$$

and assume that if $f(\theta)$ *is multiconvex that is*

$$\frac{\partial^2 f}{\partial \theta_i^2} = 2\gamma_i \geq 0, \quad i = 1, 2, \ldots, s$$

Then $f(\theta)$ *is negative in the hyper rectangle* (6) *if and only if it takes negative values at the vertices of* (6)*, that is if and only if* $f(\theta) < 0$ *for all vertices of the set given by* (6)*. For decrease the conservatism of Lemma 3 the approach proposed in [81] can be used.*

In this paragraph for uncertain gain scheduling system (3) we have proposed to use a model uncertainty in the form of a convex set with N vertices defined by (5). Furthermore, we consider the new type of performance (8) to obtain the closed-loop system guaranteed cost.

3.2 Main Results

This section formulates the theoretical approach to robust PID gain-scheduled controller design for polytopic system (3), (4), (5) which ensures closed-loop system parameter dependent quadratic stability and a guaranteed cost for all gain scheduling parameters $\theta \in \Omega_s$, and $\dot{\theta} \in \Omega_t$. The main result on robust stability for the gain-scheduled control system is given in the next theorem.

Theorem 1. *The closed-loop system* (10) *is robust parameter dependent quadratically stable with a guaranteed cost if there exist positive definite matrix* $P(\xi, \theta) \in \mathbb{R}^{n \times n}$*, matrices* N_1*,* $N_2 \in \mathbb{R}^{n \times n}$ *positive definite (semidefinite) matrices* $Q(\theta)$*,* R*,* $S(\theta)$ *and gain-scheduled controller* (9) *such that*

a)

$$\begin{aligned} L(\xi, \theta) = W_0(\xi) + \sum_{i=1}^{s} W_i(\xi)\theta_i + \\ + \sum_{i=1}^{s} \sum_{j>i}^{s} W_{ij}(\xi)\theta_i\theta_j + \sum_{i=1}^{s} W_{ii}\theta_i^2 < 0 \end{aligned} \tag{15}$$

b)

$$W_{ii}(\xi) \geq 0, \quad \theta \in \Omega_s, \; i = 1, 2, \ldots, s \tag{16}$$

where we consider parameter dependent Lyapunov matrix

$$P(\xi,\theta) = P_0(\xi) + \sum_{i=1}^{s} P_i(\xi)\theta_i > 0 \tag{17}$$

the above matrices (15) *and* (16) *are given as follows:*

$$\begin{array}{rcl}
W_0(\xi) & = & \left[\begin{array}{cc} W_{11\,0}(\xi) & W_{12\,0}(\xi) \\ * & W_{22\,0}(\xi) \end{array}\right] \\
W_{11\,0}(\xi) & = & S_0 + C_d^T F_{d0}^T R F_{d0} C_d \\
& & +N_1^T \left(I - B_0(\xi)F_{d0}C_d\right) \\
& & +\left(I - B_0(\xi)F_{d0}C_d\right)^T N_1 \\
W_{12\,0}(\xi) & = & -N_1^T \left(A_0(\xi) + B_0(\xi)F_0 C\right) \\
& & +\left(I - B_0(\xi)F_{d0}C_d\right)^T N_2 + P_0(\xi) \\
& & +C_d^T F_{d0}^T R F_0 C \\
W_{22\,0}(\xi) & = & -N_2^T \left(A_0(\xi) + B_0(\xi)F_0 C\right) \\
& & -\left(A_0(\xi) + B_0(\xi)F_0 C\right)^T N_2 + Q_0 \\
& & +C^T F_0^T R F_0 C + \sum_{j=1}^{s} P_j(\xi)\dot{\theta}_i \\
W_i(\xi) & = & \left[\begin{array}{cc} W_{11\,i}(\xi) & W_{12\,i}(\xi) \\ * & W_{22\,i}(\xi) \end{array}\right] \\
W_{11\,i}(\xi) & = & S_i + C_d^T \left(F_{d0}^T R F_{di} + F_{di}^T R F_{d0}\right) C_d \\
& & -N_1^T \left(B_0(\xi)F_{di} + B_i(\xi)F_{d0}\right) C_d \\
& & -\left[\left(B_0(\xi)F_{di} + B_i(\xi)F_{d0}\right) C_d\right]^T N_1 \\
W_{12\,i}(\xi) & = & -N_1^T \left(A_i(\xi) + B_0(\xi)F_i + B_i(\xi)F_0\right) C \\
& & -\left(B_i(\xi)F_{d0}C_d\right)^T N_2 + P_i(\xi) \\
& & +C_d^T \left(F_{di}^T R F_0 + F_{d0}^T R F_i\right) C \\
W_{22\,i}(\xi) & = & -N_2^T \left(A_i(\xi) + \left(B_0(\xi)F_i + B_i(\xi)F_0\right) C\right) \\
& & -\left[A_i(\xi) + \left(B_0(\xi)F_i + B_i(\xi)F_0\right) C\right]^T N_2 \\
& & +Q_i + C^T \left(F_0^T R F_i + F_i^T R F_0\right) C \\
W_{ij}(\xi) & = & \left[\begin{array}{cc} W_{11\,ij}(\xi) & W_{12\,ij}(\xi) \\ * & W_{22\,ij}(\xi) \end{array}\right] \\
W_{11\,ij}(\xi) & = & C_d^T \left(F_{di}^T R F_{dj} + F_{dj}^T R F_{di}\right) C_d \\
& & -N_1^T \left(B_i(\xi)F_{dj} + B_j(\xi)F_{di}\right) C_d \\
& & -C_d^T \left(B_i(\xi)F_{dj} + B_j(\xi)F_{di}\right)^T N_1 \\
W_{12\,ij}(\xi) & = & -N_1^T \left(B_i(\xi)F_j + B_j(\xi)F_i\right) C \\
& & -C_d^T \left(B_i(\xi)F_{dj} + B_j(\xi)F_{di}\right)^T N_2 \\
& & +C_d^T \left(F_{di}^T R F_j + F_{dj}^T R F_i\right) C \\
W_{22\,ij}(\xi) & = & -N_2^T \left(B_i(\xi)F_j + B_j(\xi)F_i\right) C \\
& & -C^T \left(B_i(\xi)F_j + B_j(\xi)F_i\right)^T N_2 \\
& & +C^T \left(F_i^T R F_j + F_j^T R F_i\right) C \\
W_{ii}(\xi) & = & \left[\begin{array}{cc} W_{11\,ii}(\xi) & W_{12\,ii}(\xi) \\ * & W_{22\,ii}(\xi) \end{array}\right] \\
W_{11\,ii}(\xi) & = & C_d^T F_{di}^T R F_{di} C_d - N_1^T B_i(\xi) F_{di} C_d \\
& & -C_d^T F_{di}^T B_i(\xi)^T N_1
\end{array}$$

$$
\begin{aligned}
W_{12ii}(\xi) &= -N_1^T B_i(\xi) F_i C - C_d^T F_{di}^T B_i^T(\xi) N_2 \\
&\quad + C_d^T F_{di}^T R F_i C \\
W_{22ii}(\xi) &= -N_2^T B_i(\xi) F_i C - C^T F_i^T B_i^T(\xi) N_2 \\
&\quad + C^T F_i^T R F_i C
\end{aligned}
$$

The proof is based on *Lemma 2* and *3*. The time derivative of the Lyapunov function $V(\xi,\theta) = x^T P(\xi,\theta) x$ is

$$
\frac{dV(\xi,\theta)}{dt} = \begin{bmatrix} \dot{x}^T & x^T \end{bmatrix} \begin{bmatrix} 0 & P(\xi,\theta) \\ P(\xi,\theta) & P(\xi,\dot{\theta}) \end{bmatrix} \begin{bmatrix} \dot{x} \\ x \end{bmatrix} \tag{18}
$$

where

$$
P(\xi,\dot{\theta}) = \sum_{i=1}^{s} P_i(\xi)\dot{\theta}
$$

To isolate two matrices (system and Lyapunov) introducing matrices N_1, N_2 in the following way

$$
\left[2N_1\dot{x} + 2N_2 x\right]^T \left[M_d(\xi\theta)\dot{x} - A_c(\xi,\theta)\right] = 0 \tag{19}
$$

and substituting (19), (18), $J(t)$ (8) and control law (9) to (13), after some manipulation one obtains

$$
B_e(\xi,\theta) = \begin{bmatrix} \dot{x}^T & x^T \end{bmatrix} \begin{bmatrix} W_{11}(\xi) & W_{12}(\xi) \\ W_{12}{}^T(\xi) & W_{22}(\xi) \end{bmatrix} \begin{bmatrix} \dot{x} \\ x \end{bmatrix} \tag{20}
$$

where

$$
\begin{aligned}
W_{11} &= S(\theta) + C_d^T F_d^T(\theta) R F_d(\theta) C_d + N_1^T M_d(\xi,\theta) \\
&\quad + M_d^T(\xi,\theta) N_1 \\
W_{12} &= -N_1^T A_c(\xi,\theta) + M_d^T(\xi,\theta) N_2 + P(\xi,\theta) \\
&\quad + C_d^T F_d^T(\theta) R F(\theta) C \\
W_{22} &= -N_2^T A_c(\xi,\theta) - A_c^T(\xi,\theta) N_2 + Q(\theta) \\
&\quad + C^T F^T(\theta) R F(\theta) C + P(\xi,\dot{\theta})
\end{aligned}
$$

Eq. (20) immediately implies (15), which proves the sufficient conditions of *Theorem 1*.

Eq.'s (15) and (16) are linear with respect to uncertain parameter ξ_j, $j = 1,2,\ldots,N$, therefore (15) and (16) have to hold for all $j = 1,2,\ldots,N$. For the known gain-scheduled controller parameters, inequalities (15) and (16) reduce to LMI, for gain-scheduled controller synthesis problem (15) (16) are BMI.

Remark 1. *Theorem 1* can be used for a quadratic stability test, where Lyapuunov function matrices (matrix) are either independent of parameter ξ_j, $j = 1,2,\ldots,N$ or parameter θ_i, $i = 1,2,\ldots,s$ or both as listed below.

1. Quadratic stability with respect to model parameter variation. For this case one has $P(\theta) = P_0 + \sum_{i=1}^{s} P_i\theta_i$. This Lyapunov function should withstand arbitrarily fast model parameter variation in the convex set (5)

2. Quadratic stability with respect to gain-scheduled parameters θ. For this case $P_i \rightarrow 0$, $i = 1, 2 \ldots, s$ and Lyapunov matrix is $P(\xi, \theta) = P_0(\xi)$. This Lyapunov function should withstand arbitrarily fast θ parameter variations.

3. Quadratic stability with respect to both ξ and θ parameters. Lyapunov matrix is $P(\xi, \theta) = P_0$ and it should withstands arbitrarily fast model and gain-scheduled parameter variation.

4 Robust Switched Controller Design for Nonlinear Continuous Systems

A novel approach is presented to robust switched controller design for nonlinear continuous-time systems under an arbitrary switching signal using the gain scheduling approach. The proposed design procedure is based on the robust multi parameter dependent quadratic stability condition. The obtained switched controller design procedure for nonlinear continuous-time systems is in the bilinear matrix inequality form (BMI). In the paper several forms of parameter dependent/quadratic Lyapunov functions are proposed.

4.1 Problem statement and preliminaries

4.1.1 Uncertain LPV plant model for switched systems

Consider family of nonlinear switched systems

$$\begin{aligned} \dot{z} &= f_\sigma(z, v, w) \quad \sigma \in S = \{1, 2, \ldots, N\} \\ \overline{y} &= h(z) \end{aligned} \tag{21}$$

where $z \in \mathbb{R}^n$ is the state, the input $v \in \mathbb{R}^m$, the output $\overline{y} \in \mathbb{R}^l$, exogenous input $w \in \mathbb{R}^k$ which captures parametric dependence of the plant (21) on exogenous input. The arbitrary switching algorithm $\sigma \in S$ is a piecewise constant, right continuous function which specifies at each time the index of the active system, [82]. Assume that f(.) is locally Lipschitz for every $\sigma \in S$. Consider that the number of equilibrium points for each switching modes is equal to p, that is for each mode $\sigma \in S$ the nonlinear system can be replaced by a family of p linearized plant. For more details how to obtain the gain-scheduled plant model see excellent surveys [9], [10]. To receive the model uncertainty of the gain-scheduled plant it is necessary to obtain other family of linearized plant models around the p equilibrium points. Finally, one obtains the gain-scheduled uncertain plant model in the form

$$\begin{aligned} \dot{x} &= \overline{A}_\sigma(\xi, \theta)x + \overline{B}_\sigma(\xi, \theta)u \quad \sigma \in S \\ y &= Cx \end{aligned} \tag{22}$$

where $x = z - z_e$, $u = v - v_e$, $y = \overline{y} - \overline{y}_e$, $(z_e, v_e, \overline{y}_e)$ define the equilibrium family for plant (21). Assume, that for $i - th$ equilibrium point one obtain the sets $x \in X_i$, $u \in U_i$, $y \in Y_i$, $i = 1, 2, \ldots, p$. Summarizing above sets we get $x \in X = \bigcup_{i=1}^{p} X_i$, $u \in U = \bigcup_{i=1}^{p} U_i$, $y \in Y = \bigcup_{i=1}^{p} Y_i$.

$$\begin{aligned} \overline{A}_\sigma(\xi, \theta) &= A_{\sigma 0}(\xi) + \sum_{j=1}^{p} A_{\sigma j}(\xi)\theta_j \in \mathbb{R}^{n \times n} \\ \overline{B}_\sigma(\xi, \theta) &= B_{\sigma 0}(\xi) + \sum_{j=1}^{p} B_{\sigma j}(\xi)\theta_j \in \mathbb{R}^{n \times m} \end{aligned} \tag{23}$$

Matrices $A_{\sigma j}(\xi)$, $B_{\sigma j}(\xi)$, $j = 0, 1, 2, \ldots, p$ belong to the convex set a polytope with K vertices that can formally defined as

$$\begin{aligned} \Omega_\sigma = \Bigg\{ & A_{\sigma j}(\xi), B_{\sigma j}(\xi) = \sum_{i=1}^{K} (A_{\sigma ij}, B_{\sigma ij})\xi_i \\ & j = 0, 1, 2, 3, \ldots, p, \sum_{i=1}^{K} \xi_i = 1, \xi_i \geq 0, \xi_i \in \Omega_\xi \Bigg\} \end{aligned} \tag{24}$$

where ξ_i, $i = 1, 2, \ldots, K$ are constant or possible time-varying but unknown parameters, $A_{\sigma ij}$, $B_{\sigma ij}$, C are constant matrices of corresponding dimensions, $\theta \in \mathbb{R}^p$ is a vector of known constant or time-varying gain-scheduled parameter. Assume that both lower and upper bounds are available, that is

$$\begin{aligned} \theta \in \Omega_s &= \{\theta \in \mathbb{R}^p : \theta_j \in \langle \underline{\theta}_j, \overline{\theta}_j \rangle\} \\ \dot{\theta} \in \Omega_t &= \{\dot{\theta} \in \mathbb{R}^p : \dot{\theta}_j \in \langle \underline{\dot{\theta}}_j, \overline{\dot{\theta}}_j \rangle\} \end{aligned} \tag{25}$$

4.1.2 Problem formulation

For each plant mode consider the uncertain gain-scheduled LPV plant model in the form (22),(23) and (24)

$$\begin{aligned} \dot{x} &= \left(A_{\sigma 0}(\xi) + \sum_{j=1}^{p} A_{\sigma j}(\xi)\theta_j \right) x + \left(B_{\sigma 0}(\xi) + \sum_{j=1}^{p} B_{\sigma j}(\xi)\theta_j \right) u \\ y &= Cx \end{aligned} \tag{26}$$

For a robust gain-scheduled I part controller design, the states x of (26) need to be extended in such a way that a static output feedback control algorithm can provide proportional (P) and integral (I) parts of the designed controller, for more detail see [79]. Assume that system (26) allows PI controller design with a static output feedback. The feedback control law is considered in the form

$$u = F_\sigma(\theta)y = \left(F_{\sigma 0} + \sum_{j=1}^{p} F_{\sigma j}\theta_j \right) Cx \tag{27}$$

where $F_\sigma(\theta)$ is the static output feedback gain-scheduled controller for mode σ. The closed loop system is

$$\dot{x} = A_{\sigma c}(\xi, \theta, \alpha)x \tag{28}$$

where

$$A_{\sigma c}(\xi, \theta, \alpha) =$$

$$\sum_{\sigma=1}^{N} \left(\overline{A}_\sigma(\xi, \theta) + \overline{B}_\sigma(\xi, \theta) F_\sigma(\theta) C\right) \alpha_\sigma = \sum_{\sigma=1}^{N} A_\sigma(\xi, \theta)\alpha_\sigma$$

$$\alpha^T = [\alpha_1, \alpha_2, ...\alpha_N], \ \sum_{\sigma=1}^{N} \alpha_\sigma = 1, \ \sum_{\sigma=1}^{N} \dot{\alpha}_\sigma = 0$$

$\alpha_j = 1$ when σ_j is active plant mode, else $\alpha_j = 0$. Assume $\alpha \in \Omega_\alpha$, $\dot{\alpha} \in \Omega_d$. To access the system performance, we consider an original weighted scheduled quadratic cost function

$$J = \int_{t=0}^{\infty} J(t)dt \tag{29}$$

where $J(t) = x^T Q(\theta)x + u^T Ru$, and

$$Q(\theta) = Q_0 + \sum_{j=1}^{p} Q_j \theta_j, \quad Q_j \geq 0, \quad R > 0$$

.

Definition 3. Consider a stable closed loop switched system (28) with N modes. If there is a control algorithm (27) and a positive scalar J^* such that the closed loop cost function (29) satisfies $J \leq J^*$ for all $\theta \in \Omega_s, \alpha \in \Omega_\alpha$, then J^* is said to be a guaranteed cost and "u" is said to be a guaranteed cost control algorithm for arbitrary switching algorithm $\sigma \in S$.

Theorem 2. *[80] Control algorithm (27) is the guaranteed cost control law for the switched closed loop system (28) if and only if there is Lyapunov function $V(x, \xi, \theta, \alpha) > 0$, matrices $Q(\theta), R$ and gain matrices $F_{\sigma k}$; $k = 0, 1, \ldots, p$ such that for $\sigma \in S$ the following inequality holds*

$$B_e = \frac{dV(x, \xi, \theta, \alpha)}{dt} + J(t) \leq -\varepsilon x^T x, \varepsilon \to 0 \tag{30}$$

4.2 Main results

This section formulates the theoretical approach to the robust switched gain-scheduled controller design with control law (27) which ensure closed loop multi parameter dependent quadratic stability and guaranteed cost for an arbitrary switching algorithm $\sigma \in S$. Assume that in Theorem 2 the Lyapunov function is in the form

$$V(x, \xi, \theta, \alpha) = x^T P(\xi, \theta, \alpha)x \tag{31}$$

where the Lyapunov multi parameter dependent matrix is

$$P(\xi,\theta,\alpha) = \sum_{i=1}^{K}\left(P_{0i} + \sum_{\sigma=1}^{N}(P_{\sigma 0i} + \sum_{j=1}^{p} P_{\sigma ij}\theta_j)\alpha_\sigma\right)\xi_i \tag{32}$$

Time derivative of the Lyapunov function(31) is

$$\dot{V}(.) = [\dot{x}^T \quad x^T]\begin{bmatrix} 0 & P(\xi,\theta,\alpha) \\ P(\xi,\theta,\alpha) & \dot{P}(\xi,\theta,\alpha) \end{bmatrix}\begin{bmatrix} \dot{x} \\ x \end{bmatrix} \tag{33}$$

where

$$\dot{P}(.) = \sum_{i=1}^{K}\sum_{\sigma=1}^{N} DP_{\sigma i}\alpha_\sigma \xi_i \tag{34}$$

$$DP_{\sigma i} = \sum_{\sigma=1}^{N} P_{\sigma 0i}\dot{\alpha}_\sigma + \sum_{j=1}^{p} P_{\sigma ij}\dot{\theta}_j + \sum_{j=1}^{p}\sum_{\sigma=1}^{N} P_{\sigma ij}\dot{\alpha}_\sigma\theta_j$$

Using equality

$$(2N_1\dot{x} + 2N_2 x)^T\left(\dot{x} - \sum_{\sigma=1}^{N} A_\sigma(\xi,\theta)\alpha_\sigma x\right) = 0 \tag{35}$$

equation (33) can be rewritten as

$$\frac{dV(.)}{dt} = \sum_{\sigma=1}^{N}\left[\begin{array}{cc} \dot{x}^T & x^T \end{array}\right] L_\sigma(\xi,\theta)\begin{bmatrix} \dot{x} \\ x \end{bmatrix} \tag{36}$$

$$L_\sigma(\xi,\theta) = \{l_\sigma(i,j)\}_{2\times 2}$$
$$l_\sigma(1,1) = N_1^T + N_1$$
$$l_\sigma(1,2) = -N_1^T A_\sigma(\xi,\theta) + N_2 + \sum_{i=1}^{K}\left(P_{0i} + P_{\sigma 0i} + \sum_{j=1}^{p} P_{\sigma ij}\theta_j\right)\xi_i$$

$$l_\sigma(2,2) = -N_2^T A_\sigma(\xi,\theta) - A_\sigma^T(\xi,\theta)N_2 + \sum_{i=1}^{K} DP_{\sigma i}\xi_i$$

where $N_1, N_2 \in \mathbb{R}^{n\times n}$ are auxiliary matrices.
On substituting (27) to (29) one obtains

$$J(t) = x^T S(\theta)x \tag{37}$$

where

$$S(\theta) = S_0 + \sum_{j=1}^{p} S_j\theta_j + \sum_{j=1}^{p}\sum_{k>j}^{p} S_{kj}\theta_k\theta_j + \sum_{k=1}^{p} S_{kk}\theta_k^2$$

$$S_0 = Q_0 + C^T F_{\sigma 0}^T R F_{\sigma o} C$$

$$S_j = Q_j + C^T(F_{\sigma 0}^T R F_{\sigma_j} + F_{\sigma j}^T R F_{\sigma 0})C$$
$$S_{jk} = C^T(F_{\sigma j}^T R F_{\sigma k} + F_{\sigma k}^T R F_{\sigma j})C$$
$$S_{kk} = C^T F_{\sigma k}^T R F_{\sigma k} C$$

The switched model plant (28) can be rewritten to the form

$$A_\sigma(\xi, \theta) = M_0(\xi) + \sum_{j=1}^{p} M_j(\xi)\theta_j + \sum_{j=1}^{p}\sum_{k>j}^{p} M_{jk}(\xi)\theta_j\theta_k + \tag{38}$$
$$\sum_{k}^{p} M_{kk}(\xi)\theta^2$$

where

$$M_0(\xi) = A_{\sigma 0}(\xi) + B_{\sigma 0}(\xi)F_{\sigma 0}C$$
$$M_j(\xi) = A_{\sigma j} + (B_{\sigma 0}(\xi)F_{\sigma j} + B_{\sigma j}(\xi)F_{\sigma 0})C$$
$$M_{jk}(\xi) = (B_{\sigma j}(\xi)F_{\sigma k} + B_{\sigma k}(\xi)F_{\sigma j})C$$
$$M_{kk}(\xi) = B_{\sigma k}(\xi)F_{\sigma k}C$$

Due to *Theorem 2* the closed loop switched gain-scheduled system is multi parameter dependent quadratically stable with guaranteed cost for $\sigma \in S$, ξ_i, $i = 1, 2, \ldots, K$ if the following inequalities hold

$$B_e = [\dot{x}^T \quad x^T]W(\xi, \sigma, \theta)[\dot{x}^T \quad x^T]^T \leq 0 \tag{39}$$

where $W(\xi, \sigma, \theta) = \{w_{ij}(\sigma, \xi)\}_{2\times 2}$

$$\begin{aligned}
w_{11}(\sigma, \xi) &= N_1^T + N_1 \\
w_{12}(\sigma, \xi) &= \sum_{i=1}^{K} \left(P_{0i} + P_{\sigma 0i} + \sum_{j=1}^{p} P_{\sigma ij}\theta_j \right)\xi_i \\
&- N_1^T A_\sigma(\xi, \theta) + N_2 \\
w_{22}(\sigma, \xi) &= -N_2^T A_\sigma(\xi, \theta) - A_\sigma(\xi, \theta)^T N_2 \\
&+ \sum_{i=1}^{K} DP_{\sigma i}\xi_i + S(\theta)
\end{aligned}$$

Inequality (39) implies :
- for all $\sigma \in S$ the inequality is linear with respect to uncertain parameter ξ_i, $i = 1, 2, \ldots, K$,
- for all $\sigma \in S$ the inequality is a quadratic function with respect to the gain-scheduled parameters θ_i, $i = 1, 2, \ldots, p$.
For the next development the following theorem is useful.

Theorem 3. *[81] Consider a scalar quadratic function of* $\theta \in \mathbb{R}^p$

$$f(\theta) = a_0 + \sum_{j=1}^{p} a_j\theta_j + \sum_{j=1}^{p}\sum_{k>j}^{p} a_{jk}\theta_j\theta_k + \sum_{k}^{p} a_{kk}\theta_k^2 \tag{40}$$

and assume that if $f(\theta)$ is multiconvex, that is

$$\frac{\delta^2 f(\theta)}{\delta\theta_k^2} = 2a_{kk} \geq 0,\ k = 1, 2, \ldots, p$$

then $f(\theta)$ is negative definite in the hyper rectangle (25) if and only if it takes negative values at the vertices of (25), that is if and only if $f(\theta) < 0$ for all vertices of the set given by (25).

Due to (34), (37) and (38) the robust stability conditions of switched system can be rewritten as

$$\begin{aligned} W(\xi,\sigma,\theta) = \sum_{\sigma=1}^{N} L(\theta,\xi)\alpha_\sigma = \sum_{\sigma=1}^{N}(W_{\sigma 0}(\xi)+ \\ + \sum_{j=1}^{p} W_{\sigma j}(\xi)\theta_j + \sum_{j=1}^{p}\sum_{k>j}^{p} W_{\sigma jk}(\xi)\theta_j\theta_k+ \\ + \sum_{k=1}^{p} W_{\sigma kk}\theta_k^2)\alpha_\sigma \leq 0 \end{aligned} \tag{41}$$

where $W_{\sigma 0}(\xi) = \{w_{0ij}^\sigma\}_{2\times 2}$, $W_{\sigma j}(\xi) = \{w_{jik}^\sigma\}_{2\times 2}$

$$\begin{aligned} w_{011}^\sigma &= N_1^T + N_1 \\ w_{012}^\sigma &= -N_1^T M_0(\xi) + N_2 + \textstyle\sum_{i=1}^{K}(P_{0i} + P_{\sigma 0i})\xi_i \\ w_{022}^\sigma &= -N_2^T M_0(\xi) - M_0^T(\xi)N_2 + S_0+ \\ &+ \textstyle\sum_{i=1}^{K}\left(P_{\sigma 0i}\dot{\alpha}_\sigma + \sum_{j=1}^{p} P_{\sigma ij}\dot{\theta}_j\right)\xi_i \\ w_{j11}^\sigma &= 0;\ w_{j12}^\sigma = -N_1^T M_j(\xi) + \textstyle\sum_{i=1}^{K} P_{\sigma ij}\xi_i \\ w_{j22}^\sigma &= -N_2^T M_j(\xi) - M_j(\xi)^T N_2 + S_j+ \\ &+ \textstyle\sum_{i=1}^{K}\left(\sum_{\sigma=1}^{N} P_{\sigma ij}\dot{\alpha}_\sigma\right)\xi_i \end{aligned}$$

$$W_{\sigma jk}(\xi) = \begin{bmatrix} 0 & -N_1^T M_{jk}(\xi) \\ \text{"}*\text{"} & -N_2^T M_{jk}(\xi) - M_{jk}(\xi)^T N_2 + S_{jk} \end{bmatrix}$$

$$W_{\sigma kk}(\xi) = \begin{bmatrix} 0 & -N_1^T M_{kk}(\xi) \\ \text{"}*\text{"} & -N_2^T M_{kk}(\xi) - M_{kk}(\xi)^T N_2 + S_{kk} \end{bmatrix}$$

The main results on the robust stability condition for the switched gain-scheduled control system is given in the next theorem.

Theorem 4. *Closed loop switched system (28) is robust multi parameter dependent quadratically stable with guaranteed cost if there is a positive definite matrix $P(\xi,\theta,\alpha) \in \mathbb{R}^{n\times n}$ (32), matrices $N_1, N_2 \in \mathbb{R}^{n\times n}$, positive definite (semidefinite) matrices $Q(\theta), R$ and gain-scheduled controller matrix $F_\sigma(\theta)$, such that for $\sigma \in S$*

1.

$$L_\sigma(\xi,\theta) < 0 \tag{42}$$

2.

$$W_{\sigma kk} \geq 0,\ \sigma \in S,\ \theta \in \Omega_s,\ k = 1, 2, \ldots, p$$

The proof of theorem sufficient conditions immediately follows from eqs. (32)-(41).
Notes.

- $L_\sigma(\theta, \xi)$ is linear with respect to uncertain parameter $\xi_i,\ i = 1, 2, \ldots, K$, it holds $L_\sigma(\theta, \xi) = \sum_{i=1}^{K} L_{\sigma i}(\theta)\xi_i$, therefore inequality (42) for each $\sigma \in S$ split to K inequalities of type $L_{\sigma i}(\theta) < 0$ and $W_{\sigma ikk} \geq 0$.

- Eq. (32) implies that in dependence on the chosen structure of the Lyapunov matrix $P(\xi, \theta, \alpha)$ one should obtained different types of stability conditions from quadratic to multi parameter dependent quadratic stabilities. Different types of stability conditions determine the conservatism of the design procedure and the rate of change of corresponding variables.

5 Gain-Scheduled MPC Design for Nonlinear Systems with Input Constraints

A novel methodology is proposed for discrete model predictive gain-scheduled controller design for nonlinear systems with input(hard)/output(soft) constraints for finite and infinite prediction horizons. The proposed design procedure is based on the linear parameter-varying (LPV) paradigm, affine parameter-dependent quadratic stability and on the notion of the parameter-varying guaranteed cost. The design procedure is in the form of BMI (we can use a free and open source BMI solver).

5.1 Problem formulation and preliminaries

Consider a nonlinear plant $x(k+1) = F(x(k), u(k), \theta(k))$ which is identified in several working points. The identified family of linear systems in discrete-time space is given as follows

$$\begin{aligned} x(k+1) &= A_i\, x(k) + B_i\, u(k) \\ y(k) &= C_i\, x(k) \quad i = 1, 2, \ldots, N \end{aligned} \tag{43}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m$ is the controller output, $y(k) \in \mathbb{R}^l$ is the measured plant output vector at step $k \in \mathbb{R}_+$, matrices $A_i,\ B_i,\ C_i$, $i = 1, 2, \ldots, N$ are system matrices with appropriate dimension and N is the number of identified plants model. Assume that a known vector $\theta(k) \in \Omega$ exists which captures the parametric dependence of the linearized model (43) on the equilibrium (working) points of the original nonlinear system.

5.1.1 Case of finite prediction horizon

The identified family of linear systems (43) for a given prediction and control horizon N_k can be transformed to the following form [83]

$$\begin{aligned} z(k+1) &= A_{f\,i}\, z(k) + B_{f\,i}\, v(k) \\ y_f(k) &= C_{f\,i}\, z(k) \quad i = 1, 2, \ldots, N \end{aligned} \tag{44}$$

where

$$\begin{aligned} z^T(k) &= \left[x^T(k|k)\; x^T(k+1|k) \cdots x^T(k+N_k-1|k) \right] \\ v^T(k) &= \left[u^T(k|k)\; u^T(k+1|k) \cdots u^T(k+N_k-1|k) \right] \\ y_f^T(k) &= \left[y^T(k|k)\; y^T(k+1|k) \cdots y^T(k+N_k-1|k) \right] \end{aligned}$$

$$A_{f\,i} = \begin{bmatrix} A_i & 0 \cdots 0 \\ A_i^2 & 0 \cdots 0 \\ \vdots & \vdots \ddots 0 \\ A_i^{N_k} & 0 \cdots 0 \end{bmatrix}, \quad C_{f\,i} = \begin{bmatrix} C_i & 0 & \cdots & 0 \\ 0 & C_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_i \end{bmatrix}$$

$$B_{f\,i} = \begin{bmatrix} B_i & 0 & \cdots & 0 \\ A_i B_i & B_i & \cdots & 0 \\ A_i^2 B_i & A_i B_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_i^{N_k-1} B_i & A_i^{N_k-2} B_i & \cdots & B_i \end{bmatrix}$$

and $x(k+j|k)$ is the step ahead prediction of the state, calculated in sample time k. From the family of linear systems (44) one obtains [84] a gain-scheduled plant model in the form

$$\begin{aligned} z(k+1) &= A_{f\,a}(\theta(k))\, z(k) + B_{f\,a}(\theta(k))\, v(k) \\ y_f(k) &= C_{f\,a}\, z(k) \end{aligned} \tag{45}$$

where $C_{f\,a} = C_{f\,1} = C_{f\,2} = \ldots = C_{f\,N}$ and

$$\begin{aligned} A_{f\,a}(\theta(k)) &= A_{f\,a0} + \sum_{i=1}^{p} A_{f\,ai} \theta_i(k) \\ B_{f\,a}(\theta(k)) &= B_{f\,a0} + \sum_{i=1}^{p} B_{f\,ai} \theta_i(k) \end{aligned}$$

and $A_{f\,ai}$, $B_{f\,ai}$, $i = 0, 1, \ldots, p-1$ are system matrices with appropriate dimension, $A_{f\,ap} = 0$, $B_{f\,ap} = 0$, $\theta(k)^T = [\theta_1(k), \theta_2(k), \ldots, \theta_{p-1}(k)] \in \Omega$ is the vector of $p-1$ known independent scheduling parameters at step k and $\theta_p \in \langle 0, H_m \rangle$ is the scheduled parameter which is used to ensure I/O constraints, where $H_m \in (0, 1)$. The control law for the model predictive gain-scheduled controller design for a given prediction and control horizon N_k is considered in the form

$$v(k) = F(\theta(k))\, y_f(k) = F(\theta(k))\, C_{f\,a}\, z_f(k) \tag{46}$$

where $F(\theta(k)) = F_0 + \sum_{j=1}^{p-1} F_j \theta_j(k) - F_0 \theta_p(k)$.

Note 1. We can extend system (45) to PS or PSD control, for more information see [85].

The procedure to ensure the input/output constraints is very simple. If the system input or output approach the maximal or minimal value, using the scheduling parameter θ_p one can affect the controller output. There are several solutions how to generate the scheduling parameter θ_p, it is depending on the system. We will deal with this issue in the examples. If we substitute control law (46) to system (45), a closed-loop system is obtained

$$z(k+1) = A_c(\theta(k))z(k) \tag{47}$$

where $A_c(\theta(k)) = A_{f\,a}(\theta(k)) + B_{f\,a}(\theta(k))F(\theta(k))C_{f\,a}$.

To assess the performance quality with possibility to obtain different performance quality in each working point a quadratic cost function described in paper [85] will be used

$$\begin{aligned} J_{d\,f}(\theta(k)) = \sum_{k=0}^{\infty} & z_f(k)^T Q(\theta(k)) z_f(k) + v(k)^T R v(k) \\ & + \Delta z_f(k)^T S(\theta(k)) \Delta z_f(k) = \sum_{k=0}^{\infty} J_d(\theta(k)) \end{aligned} \tag{48}$$

where $\Delta z_f(k) = z_f(k+1) - z_f(k)$, $Q(\theta(k)) = Q_0 + \sum_{i=1}^{p} Q_i\theta_i(k)$, $S(\theta(k)) = S_0 + \sum_{i=1}^{p} S_i\theta_i(k)$, $Q_i = Q_i^T \geq 0$, $S_i = S_i^T \geq 0$, $R > 0$ and $Q_p = S_p = 0$.

Note 2. Using the cost function (48) we can affect the performance quality separately in each working point with defining different weighting matrices for each working point which then are transformed to affine form and depend on the scheduled parameters as system matrices. [85]

Definition 4. Consider system (45) with control algorithm (46). If a control law v^* and a positive scalar J_d^* exist such that the closed-loop system (47) is stable and the value of closed-loop cost function (48) satisfies $J_d \leq J_d^*$, then J_d^* is said to be a guaranteed cost and v^* is said to be guaranteed cost control law for system (45).

Substituting the control law (46) to the quadratic cost function (48) one can obtain

$$J_d(\theta(k)) = \tilde{z}^T \begin{bmatrix} J_{d11}(\theta(k)) & J_{d12}(\theta(k)) \\ J_{d12}^T(\theta(k)) & J_{d22}(\theta(k)) \end{bmatrix} \tilde{z} \tag{49}$$

where $\tilde{z}^T = \begin{bmatrix} z^T(k+1) & z^T(k) \end{bmatrix}$ and

$$\begin{aligned} J_{d11}(\theta(k)) &= S(\theta(k)), \quad J_{d12}(\theta(k)) = -S(\theta(k)), \\ J_{d22}(\theta(k)) &= Q(\theta(k)) + C_{f\,a}^T F(\theta(k))^T R\, F(\theta(k))\, C_{f\,a} \\ &\quad + S(\theta(k)) \end{aligned}$$

To ensure the Affine Quadratic Stability (AQS) [81] the following Lyapunov function has been chosen

$$V(\theta(k)) = z_f^T(k)P(\theta(k))z_f(k) \tag{50}$$

The first difference of Lyapunov function (50) is given as follows

$$\begin{aligned}\Delta V(\theta(k)) = z_f^T(k+1)\,P(\theta(k+1))\,z_f(k+1)- \\ -\,z_f^T(k)\,P(\theta(k))\,z_f(k)\end{aligned} \tag{51}$$

where

$$P(\theta(k)) = P_0 + \sum_{i=1}^{p} P_i\theta_i(k) \tag{52}$$

On substituting $\theta(k+1) = \theta(k)+\Delta\theta(k)$ to $P(\theta(k+1))$ one obtains the following result

$$P(\theta(k+1)) = P_0 + \sum_{i=1}^{p} P_i\theta_i(k) + \sum_{i=1}^{p} P_i\Delta\theta_i(k) \tag{53}$$

where if assuming that $P_i > 0$, $\Delta\theta_i \in \langle\Delta\underline{\theta_i}, \Delta\overline{\theta_i}\rangle \in \Omega_t$, $i = 0, 1, \ldots, p$ and $\max|\Delta\theta_i| < \rho_i$, one can write

$$P(\theta(k+1)) \leq P_0 + \sum_{i=1}^{p} P_i\theta_i(k) + P_\rho = P_\rho(\theta(k)) \tag{54}$$

where $P_\rho = \sum_{i=1}^{p} P_i\rho_i$. The first difference of the Lyapunov function (51) using the free matrix weighting approach [84] is in the form

$$\Delta V(\theta(k)) = \tilde{z}^T \begin{bmatrix} V_{11}(\theta(k)) & V_{12}(\theta(k)) \\ V_{12}^T(\theta(k)) & V_{22}(\theta(k)) \end{bmatrix} \tilde{z} \tag{55}$$

where

$$\begin{aligned} V_{11}(\theta(k)) &= P_\rho(\theta(k)) + N_1 + N_1^T \\ V_{12}(\theta(k)) &= N_2^T - N_1A_c(\theta(k)) \\ V_{22}(\theta(k)) &= -P(\theta(k)) - N_2A_c(\theta(k)) - A_c^T(\theta(k))N_2^T \end{aligned}$$

where $N_1, N_2 \in \mathbb{R}^{n\times n}$ are auxiliary matrices.

Definition 5. [81] The linear closed-loop system (47) for $\theta(k) \in \Omega$ and $\Delta\theta(k) \in \Omega_t$ is affinely quadratically stable if and only if $p+1$ symmetric matrices $P_0, P_1, \ldots, P_p$ exist such that $P(\theta(k))$ (52), $P_\rho(\theta(k))$ (54) are positive defined and for the first difference of the Lyapunov function (55) along the trajectory of closed-loop system (47) it holds

$$\Delta V(\theta(k)) < 0 \tag{56}$$

From LQ theory we can introduce the well known results:

Lemma 4. *Consider the closed-loop system* (47). *Closed-loop system* (47) *is affinely quadratically stable with guaranteed cost if and only if the following inequality holds*

$$B_e(\theta(k)) = \min_u \{\Delta V(\theta(k)) + J_d(\theta(k))\} \leq 0 \tag{57}$$

for all $\theta(k) \in \Omega$*. For proof see [80].*

5.1.2 Case of infinite prediction horizon

The system described by (45) for the case of $N_k = 0$ can be transformed to the gain-scheduled plant model

$$\begin{aligned} x(k+1) &= A(\theta(k))\, x(k) + B(\theta(k))\, u(k) \\ y(k) &= C\, x(k) \end{aligned} \tag{58}$$

where $A(\theta(k)) = A_0 + \sum_{i=1}^{p} A_i\theta_i(k)$, $B(\theta(k)) = B_0 + \sum_{i=1}^{p} B_i\theta_i(k)$, $A_p = 0$ and $B_p = 0$. For the case $N_k \to \infty$ and $S = 0$ the cost function (48) can be rewritten as

$$\begin{aligned} J = \sum_{k=0}^{\infty} J(k) = \sum_{k=0}^{\infty} \Bigg(& \sum_{j=0}^{\infty} x^T(k+j) q_j x(k+j) \\ & + u^T(k+j) r_j u(k+j) \Bigg) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \tilde{J}(k) \end{aligned} \tag{59}$$

where $q_j \in \mathbb{R}^{n\times n}$, $r_j \in \mathbb{R}^{m\times m}$ are positive definite matrices. The control law for the model predictive gain-scheduled controller design for the infinite prediction horizon is considered in the form

$$u(k) = F(\theta(k))\, y(k) = F(\theta(k))\, C\, x(k) \tag{60}$$

where $F(\theta(k)) = F_0 + \sum_{j=1}^{p-1} F_j\theta_j(k) - F_0\theta_p(k)$. To guarantee the stability and performance of the closed-loop gain-scheduled system, due to *Lemma 4* it is sufficient to ensure

$$B_e(\theta(k)) = \Delta V(x(k+j), \theta(k)) + \tilde{J}(k) \leq 0 \tag{61}$$

where $\Delta V(x(k+j), \theta(k)) = V(x(k+j+1), \theta(k) + \Delta\theta(k)) - V(x(k), \theta(k))$ is the first difference of the Lyapunov function for j horizon prediction. Summing (61) from $j = 0$ to $j \to \infty$, the upper bound on $J(k)$ is obtained

$$J(k) \leq V(x(k), \theta(k)) \tag{62}$$

On the basis of (62) the following gain-scheduled MPC design procedure is given

$$\min_{F(\theta(k))} V(x(k), \theta(k)) \tag{63}$$

with constraints to system model (58), stability model and performance (61) and other constraints. Assume that the Lyapunov function is in the form $V(x(k), \theta(k)) = x^T(k)P(\theta(k))x(k)$, where $P(\theta) = P_0 + \sum_{i=1}^{p} P_i\theta_i(k)$. Due to (63), the predicted control design procedure can be modified as

$$\min_{F(\theta(k))} x^T(k)P(\theta(k))x(k) \leq x^T(k)x(k)\gamma \tag{64}$$

which leads to the inequality

$$P(\theta(k)) \leq \min_{F(\theta(k))} \gamma I \tag{65}$$

In the paper [86] inequality (64) is in the form

$$\min x^T(k)P(\theta(k))x(k) \leq \gamma \tag{66}$$

which needs to know the state vector $x(k)$ and on-line optimization of (66) at every sample time.

The stability of the closed-loop system is guaranteed if

$$B_e = \Delta V(x(k), \theta(k)) + \alpha V(x(k), \theta(k)) \leq 0 \tag{67}$$

where $\alpha \in \langle 0, 1)$ is a coefficient with an influence on the closed-loop system performance. If we substitute the Lyapunov function and its first difference to (67), we can obtain

$$B_e(\theta(k)) = \tilde{x}^T W(\theta(k))\, \tilde{x} \leq 0 \tag{68}$$

where $\tilde{x}^T = \begin{bmatrix} x^T(k+1)\, x^T(k) \end{bmatrix}$, $\beta = 1 - \alpha$ and

$$\begin{aligned}
W(\theta(k)) \quad &= \begin{bmatrix} W_{11}(\theta(k)) & W_{12}(\theta(k)) \\ W_{12}(\theta(k))^T & W_{22}(\theta(k)) \end{bmatrix} \\
W_{11}(\theta(k)) &= N_1^T + N_1 + P_\rho(\theta(k)) \\
W_{12}(\theta(k)) &= -N_1^T A_c(\theta(k)) + N_2 \\
W_{22}(\theta(k)) &= -N_2 A_c(\theta(k)) - A_c^T(\theta(k))N_2 - P(\theta(k))(\beta)
\end{aligned}$$

5.2 Main results

In this section the discrete predictive gain-scheduled controller design procedure is presented which guarantees the affine quadratic stability and guaranteed cost for $\theta(k) \in \Omega$ with pre-defined maximal rate of change of the scheduled parameters ρ. The main result of this section – the discrete model predictive gain-scheduled controller design procedure – relies on the concept of multi-convexity, that is convexity along each direction $\theta_i(k)$, $i = 1, 2, \ldots, p$ of the parameter space. The implications of multiconvexity for scalar quadratic functions are given in the next lemma [81].

Lemma 5. *Consider a scalar quadratic function of $\alpha \in \mathbb{R}^p$.*

$$f(\alpha) = a_0 + \sum_{i=1}^{p} a_i\alpha_i + \sum_{i=1}^{p}\sum_{j>i}^{p} b_{ij}\alpha_i\alpha_j + \sum_{i=1}^{p} c_i\alpha_i^2 \tag{69}$$

and assume that $f(\alpha_1, \dots, \alpha_p)$ *is multi-convex, that is* $\frac{\partial^2 f(\alpha)}{\partial \alpha_i^2} = 2c_i \geq 0$ *for* $i = 1, 2, \dots, p$. *Then* $f(\alpha)$ *is negative for all* $\alpha \in \Omega$ *and* $\dot{\alpha} \in \Omega_t$ *if and only if it takes negative values at the corners of* α.

5.2.1 Finite prediction horizon

Using *Lemmas 4* and *5* the following theorem is obtained for discrete model predictive gain-scheduled controller design for finite horizon.

Theorem 5. *Closed-loop system* (47) *is affinely quadratically stable if* $p+1$ *symmetric matrices* $P_0, P_1, \dots, P_p$ *exist such that* $P(\theta(k))$ (52), $P_\rho(\theta(k))$ (54) *are positive definite for all* $\theta(k) \in \Omega$, *with pre-defined* ρ_i, *matrices* N_1, N_2, Q_i, R, S_i, $i = 1, 2, \dots, p$ *and gain-scheduled matrices* $F(\theta(k))$ *satisfying*

$$\begin{aligned} M(\theta(k)) &< 0; \quad \theta(k) \in \Omega \\ M_{ii} &\geq 0; \quad i = 1, 2, \dots, p \end{aligned} \tag{70}$$

where (at sample time k)

$$\begin{aligned}
M(\theta) &= M_0 + \textstyle\sum_{i=1}^{p} M_i\theta_i + \sum_{i=1}^{p}\sum_{j>i}^{p-1} M_{ij}\theta_i\theta_j + \sum_{i=1}^{p} M_{ii}\theta_i^2 \\
M_0 &= \begin{bmatrix} M_{110} & M_{120} \\ M_{120}^T & M_{220} \end{bmatrix}, \ M_i = \begin{bmatrix} M_{11i} & M_{12i} \\ M_{12i}^T & M_{22i} \end{bmatrix} \\
M_{ij} &= \begin{bmatrix} M_{11ij} & M_{12ij} \\ M_{12ij}^T & M_{22ij} \end{bmatrix}, M_{110} = P_0 + N_1 + N_1^T + S_0 + P_\rho \\
M_{11i} &= P_i, \ M_{11ij} = 0, \ M_{11ii} = 0 \\
M_{120} &= N_2 - N_1^T(A_{fa0} + B_{fa0}F_0C_{fa}) - S_0 \\
M_{12i} &= -N_1^T(A_{fai} + B_{fai}F_0C_{fa} + B_{fa0}F_iC_{fa}) - S_i \\
M_{12ij} &= -N_1^T(B_{fai}F_j + B_{faj}F_i)C_{fa} \\
M_{12ii} &= -N_1^T B_{fai}F_iC_{fa} \\
M_{220} &= Q_0 + S_0 - P_0 - N_2^T(A_{fa0} + B_{fa0}F_0C_{fa}) \\
&\quad -(A_{fa0} + B_{fa0}F_0C_{fa})^T N_2 + C_{fa}^T F_0^T RF_0C_{fa} \\
M_{22i} &= -P_i - N_2^T(A_{fai} + B_{fai}F_0C_{fa} + B_{fa0}F_iC_{fa}) \\
&\quad -(A_{fai} + B_{fai}F_0C_{fa} + B_{fa0}F_iC_{fa})^T N_2 \\
&\quad +C_{fa}^T(F_0^T RF_i + F_i^T RF_0)C_{fa} + Q_i + S_i \\
M_{22ij} &= -N_2^T(B_{fai}F_j + B_{faj}F_i)C_{fa} - C_{fa}^T(B_{fai}F_j \\
&\quad +B_{faj}F_i)^T N_2 + C_{fa}^T(F_i^T RF_j + F_j^T RF_i)C_{fa} \\
M_{22ii} &= -N_2^T B_{fai}F_iC_{fa} - (B_{fai}F_iC_{fa})^T N_2 \\
&\quad +C_{fa}^T F_i^T RF_iC_{fa}
\end{aligned}$$

Proof. The proof of the *Theorem 5* is clear from the previous derivations. Here, the proof is repeated only in basic steps. The proof is based on the *Lemmas 4* and *5*. When substituting the first difference of the Lyapunov function (55) and the quadratic cost function (49) to the Bellman-Lyapunov function (57), after some manipulation, using *Lemma 5* we obtain (70) which proofs the *Theorem 5*. □

5.2.2 Infinite prediction horizon

Using inequalities (65), (68) and *Lemma 5* the following theorem is obtained for discrete model predictive gain-scheduled controller design for infinite horizon.

Theorem 6. *Closed-loop system is affinely quadratically stable if there exist $p+1$ symmetric matrices $P_0, P_1, \ldots, P_p$ such that $P(\theta(k))$ (52), $P_\rho(\theta(k))$ (54) are positive definite for all $\theta(k) \in \Omega$, with pre-defined ρ_i, matrices N_1, N_2, and gain-scheduled matrices $F(\theta(k))$ satisfying*

$$\begin{aligned} W(\theta(k)) &< 0; \quad \theta(k) \in \Omega \\ W_{ii} &\geq 0; \quad i = 1, 2, \ldots, p \\ P(\theta(k)) &\leq \min_{F(\theta(k))} \gamma \end{aligned} \tag{71}$$

where (at sample time k)

$$\begin{aligned}
W(\theta) &= W_0 + \sum_{i=1}^{p} W_i\theta_i + \sum_{i=1}^{p}\sum_{j>i}^{p-1} W_{ij}\theta_i\theta_j + \sum_{i=1}^{p} W_{ii}\theta_i^2 \\
W_0 &= \begin{bmatrix} W_{110} & W_{120} \\ W_{120}^T & W_{220} \end{bmatrix}, \quad W_i = \begin{bmatrix} W_{11i} & W_{12i} \\ W_{12i}^T & W_{22i} \end{bmatrix} \\
W_{ij} &= \begin{bmatrix} W_{11ij} & W_{12ij} \\ W_{12ij}^T & W_{22ij} \end{bmatrix}, \; W_{110} = P_0 + N_1 + N_1^T + P_\rho \\
W_{11i} &= P_i, \; W_{11ij} = 0, \; W_{11ii} = 0 \\
W_{120} &= N_2 - N_1^T(A_0 + B_0F_0C) \\
W_{12i} &= -N_1^T(A_i + B_iF_0C + B_0F_iC), \\
W_{12ij} &= -N_1^T(B_iF_j + B_jF_i)C \\
W_{220} &= -N_2^T(A_0 + B_0F_0C) - (A_0 + B_0F_0C)^T N_2 - P_0\beta \\
W_{22i} &= -N_2^T(A_i + B_iF_0C + B_0F_iC) \\
&\quad -(A_i + B_iF_0C + B_0F_iC)^T N_2 - P_i\beta \\
W_{22ij} &= -N_2^T(B_iF_j + B_jF_i)C - C^T(B_iF_j + B_jF_i)^T N_2 \\
W_{12ii} &= -N_1^T B_iF_iC, \; W_{22ii} = -N_2^T B_iF_iC - (B_iF_iC)^T N_2
\end{aligned}$$

Proof. The proof of the *Theorem 6* regarding to space limitations is sketched only in basic steps. The proof is based on the *Lemmas 4* and *5*. If we substitute the Lyapunov function and its first difference to (67), we can obtain (68), after some manipulation, using *Lemma 5* we obtain (71) which proofs the *Theorem 6*. □

6 Concluding remarks

6.1 Brief overview

This thesis deals with controller design for nonlinear systems. The controller is given in a feedback structure, that is the controller has informations about the system and use it to influence the system. The nonlinear system is transformed to linear parameter-varying system, which is used to design a controller, i.e. gain-scheduled controller. The gain-scheduled controller synthesis in this thesis is based on the Lyapunov theory of stability as well as on the Bellman-Lyapunov function. To achieve a performance quality a quadratic cost function and its modifications known from LQ theory are used. The obtained gain-scheduled controller guarantees the closed-loop stability and the guaranteed cost. The main results for controller synthesis are in the form of bilinear matrix inequalities and/or linear matrix inequalities. For controller synthesis one can use a free and open source BMI solver PenLab or LMI solvers LMILab or SeDuMi.

6.2 Research results

In the initial stage of our research I with my supervisor Prof. Ing. Vojtech Veselý, DrSc. developed a gain-scheduled controller design which guarantees the closed-loop system stability and a guaranteed cost for continuous-time linear parameter-varying (LPV) systems for all scheduled parameter changes with pre-defined rate of scheduled parameter changes. These results were published in Journal of Process Control and presented at several conferences (ICCC'13, ICPC'13, ELITECH'13, IN-TECH'13). After that we expanded this theory to robust controller design for continuous and discrete-time uncertain LPV systems with possibility for variable weighting in cost function. Some of these results were published in Journal of Electrical Engineering, in Journal of Electrical Systems and Information Technology and they were presented at several conferences such as the European Control Conference 2014 (ICCC'14, CPS'14, SSKI'14, ELITECH'14, ICPC'15).

In the middle stage of our research we modified our approaches from BMI (bilinear matrix inequality) to LMI (linear matrix inequality) problem. This caused that our controller synthesis works for high-order systems (50-60th order was tested). We successfully ported our approaches to switched and model predictive controller design. Some of these results have been published in Journal of the Franklin Institute, in Journal of Electrical Engineering, in International Review of Automatic Control, in Asian Journal of Control and will be presented at several IFAC symposiums and conferences as MICNON'15 or ROCOND'15 (ICPC'15, ELITECH'15). Also some of these results are under review process in journals International Journal of Control, Automation and Systems and in Archives of Control Sciences.

In the final stage we successfully developed a new stability condition where we could bypass the multi-convexity that significantly reduced the conservativeness of the controller synthesis. In addition we added to the controller synthesis the input (hard) / output (soft) constraints as well as input rate (soft) / output rate (soft) constraints where we do not need online optimization. Publications of these results are only in the preliminary stage but hopefully they will be published in high impact factor journals, too.

6.3 Closing remarks and future works

The main goal for this thesis (and also to our research in last 2,5 – 3 years) was to find a systematic controller design approach for uncertain nonlinear systems, which guarantees the closed-loop stability and guaranteed cost with considering input/output constraints, all this without on-line optimization and need of high-performance industrial computers.

This summary presents the selected results from 3 published papers. One can find the main results for robust gain-scheduled controller design, for robust switched controller design and for discrete gain-scheduled MPC design with input constraints. Dissertation thesis consists from 9 papers, which covers the main research results obtained within the last 2.5 years. We tried to select those publications, which best reflect the achieved results. The first included paper (*Chapter 4*) presents a simple gain-scheduled controller design for nonlinear systems, which guarantees the closed-loop stability and guaranteed cost. One can include the maximal value of the rate of gain-scheduled parameter changes, which allows to decrease conservativeness and obtain the controller with a given performance. In the next chapter one can find a simple modification of these results, where a new quadratic cost function is used, where weighting matrices are time-varying and depends on scheduled parameter. Using these original variable weighting matrices we can affect performance quality separately in each working points and we can tune the system to the desired condition through all parameter changes. *Chapters 6, 7* presents the robust versions of the obtained results from *Chapters 4, 5*, where in *Chapter 6* the design procedure is transformed from the bilinear matrix inequality form to linear matrix inequality, which caused that our controller synthesis works for high order systems. In *Chapter 8* a simplified version of the robust controller design in discrete time domain is presented, where a new LPV description of T1DM Bergman's minimal model with two additional subsystems (absorption of digested carbohydrates and subcutaneous insulin absorption) is created. The controller synthesis in this paper is also transformed to LMI problem. In *Chapters 9* and *10* a gain-scheduled controller designs adopted to switched control are presented in continuous time. In the proposed design procedures there is no need to use the notion of the "dwell-time" for arbitrary switching, which significantly simplifies the switched controller design compared to approaches in the literatures. In *Chapter 11* a novel gain scheduling based model predictive controller design procedure for nonlinear systems is presented for finite and infinite prediction horizons with considering input/output constraints. Finally a novel unified robust gain-scheduled and switched controller design approach is presented in *Chapter 12* where the conservativeness from multi-convexity is eliminated.

The stated objectives was reached successfully, but there are many unsolved problems yet. For example, in this thesis it is hypothesized, that the scheduled parameters can be measured and the measurement is accurate. It is true, that if one use the robust version, then the measurement inaccuracy can be covered as model uncertainty, but this should be studied in more detail. Furthermore, it would be good to study, how can be used the informations from disturbances to improve the performance quality under disturbances. Moreover it would be an interesting study how to reduce the time required to controller synthesis, because it is well known, that the required time for controller design using LMI and especially BMI solvers, rapidly increases for higher order systems. It follows, that this thesis opens new possibilities for further studies and research in this specific area.

List of publications

Original research papers in international scientific journals (peer reviewed)

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Notes/Comments